

# THE YOSIDA CLASS IS UNIVERSAL

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January 14, 2013

**ABSTRACT.** We discuss families of meromorphic functions  $f_h$  obtained from single functions  $f$  by the re-scaling process  $f_h(z) = h^{-\alpha} f(h + h^{-\beta} z)$  generalising Yosida's process  $f_h(z) = f(h + z)$ . The main objective is to obtain information on the value distribution of the generating functions  $f$  themselves. Among the most prominent generalised Yosida functions are first, second and fourth Painlevé transcendents. The Yosida class contains all limit functions of generalised Yosida functions—the Yosida class is universal.

**KEYWORDS.** Normal family, Nevanlinna theory, spherical derivative, Painlevé transcendents, elliptic function, Yosida function, re-scaling,

2010 MSC. 30D30, 30D35, 30D45

## 1. INTRODUCTION

**YOSIDA FUNCTIONS.** In [17] Yosida introduced the class (A) of transcendental meromorphic in the complex plane having bounded spherical derivative

$$(1) \quad f^\# = \frac{|f'|}{1 + |f|^2}.$$

Then the translates  $f_h(z) = f(z + h)$  of  $f$  in the class (A) form a normal family in  $\mathbb{C}$ , and vice versa;  $f$  is called of the *first category*, if no limit function

$$(2) \quad \mathfrak{f} = \lim_{h_n \rightarrow \infty} f_{h_n}$$

is a constant (convergence is always understood with respect to the spherical metric). It is this additional condition that makes the class  $A_0$  of *Yosida functions* so fascinating. The elementary functions (like  $e^z$ ,  $\tan z$  etc) have bounded spherical derivatives, but are not Yosida functions. On the other hand  $A_0$  contains the elliptic functions. A thorough investigation of the class  $A_0$  was performed by Favorov [3], with emphasis on the distribution of zeros and poles.

**PAINLEVÉ TRANSCENDENTS.** The *first Painlevé transcendents* are the solutions to Painlevé's first differential equation  $w'' = z + 6w^2$ ; they are meromorphic in  $\mathbb{C}$  (see [14]) and satisfy  $w^\# = O(|z|^{\frac{3}{4}})$ . More precisely, if  $\mathcal{Q}$  denotes the set of (non-zero) zeros of  $w$  and  $\mathcal{Q}_\epsilon = \bigcup_{q \in \mathcal{Q}} \{z : |z - q| < \epsilon |q|^{-\frac{1}{4}}\}$ , then  $w^\#(z) = O(|z|^{-\frac{1}{4}})$  holds outside  $\mathcal{Q}_\epsilon$ , while  $w^\#(q) \asymp |q|^{\frac{3}{4}}$ . The family  $(w_h)_{|h| \geq 1}$  with  $w_h(z) = h^{-\frac{1}{2}} w(h + h^{-\frac{1}{4}} z)$  is normal in  $\mathbb{C}$ , and for “most” solutions the limit functions  $\lim_{h_n \rightarrow \infty} w_{h_n}$  are non-constant. There exist, however, solutions with large zero- and pole-free regions, and in that case one

has constant solutions  $\neq 0, \infty$ , see [15]. We note, however, that in many applications it is only required that  $f_{h_n} \not\rightarrow 0, \infty$ .

DEFINITION. The class  $\mathcal{Y}_{\alpha,\beta}$  ( $\alpha \in \mathbb{R}$ ,  $\beta > -1$ ) consists of all in  $\mathbb{C}$  transcendental meromorphic functions  $f$  in  $\mathbb{C}$ , such that the family  $(f_h)_{|h| \geq 1}$  of functions

$$(3) \quad f_h(z) = h^{-\alpha} f(h + h^{-\beta} z)$$

(for any or just one determination of  $h^{-\alpha}$  and  $h^{-\beta}$ ) form a normal family in  $\mathbb{C}$ , and no limit function (2) is constant. Functions in  $\mathcal{Y}_{\alpha,\beta}$  are called *generalised Yosida functions*. We also define the classes  $\mathcal{Y}_{\alpha,-1}$  by postulating normality of the family of functions  $f_h(z) = h^{-\alpha} f(h + hz)$  only in  $\mathbb{C} \setminus \{-1\}$ , and postpone the analysis of this class to the last section.

REMARKS AND EXAMPLES.

- Some of the results proved in this paper are not new. This, in particular, concerns theorems in the classes  $\mathcal{W}_1^0 = \mathcal{Y}_{0,-1}$  (see the last section) and  $A_0 = \mathcal{Y}_{0,0}$ . The proofs in this paper cover all parameters  $\alpha \in \mathbb{R}$  and  $\beta > -1$ .
- The re-scaling process in the definition of  $\mathcal{Y}_{\alpha,\beta}$  is motivated by and formally related to the Pang-Zalcman process [11, 12, 19, 20]. As far as I know, particular classes  $\mathcal{Y}_{\alpha,\beta}$  with  $\alpha \neq 0$  occurred for the first time, although implicitly, in the paper [15] on the Painlevé transcendents. Of course, this kind of re-scaling is also not new. It goes back at least to Valiron, but was even used in Painlevé's so-called  $\alpha$ -method.
- It will turn out that  $\mathcal{Y}_{\alpha,\beta}$  is contained in the class  $\mathcal{W}_{2+|\alpha|+\beta}$  discussed by Gavrilov [4]:  $f \in \mathcal{W}_p$  ( $p \geq 1$ ) if and only if  $\sup_{\mathbb{C}} |z|^{2-p} f^\#(z) < \infty$ , see also Makhmutov [8];  $\mathcal{W}_2$  is Yosida's class (A). The class  $\mathcal{W}_p^{(0)}$ , also discussed by Gavrilov, coincides with  $\mathcal{Y}_{0,p-2}$ , while the same is true for  $A_0$  and  $\mathcal{Y}_{0,0}$ .
- $f \in \mathcal{Y}_{\alpha,\beta}$  implies  $1/f \in \mathcal{Y}_{-\alpha,\beta}$ , and  $\tilde{f}(z) = z^a f(z^b)$  ( $a \in \mathbb{Z}, b \in \mathbb{N}$ ) belongs to  $\mathcal{Y}_{a+b\alpha, b+b\beta-1}$ ;  $z^b$  and  $z^a$  may be replaced by a polynomial  $p$  and a rational function  $r$ , respectively, with  $\deg p = b$  and  $r(z) \sim cz^a$  as  $z \rightarrow \infty$  ( $c \neq 0$ ). We mention two simple corollaries:
  - If  $\alpha = -a/b$  is rational, then  $\tilde{f} \in \mathcal{Y}_{0, \beta+b\beta-1}$ .
  - If  $-1 < \beta < 0$  and  $b$  is sufficiently large, then  $b + b\beta - 1 \geq 0$ .

It would thus suffice to deal with the cases  $\beta = -1$  and  $\beta \geq 0$ , respectively.

- To every  $n \in \{2, 3, 4, 6\}$  there exists a meromorphic function  $f$  such that  $f(z^n)$  is an elliptic function (see Mues [9]). Thus  $f \in \mathcal{Y}_{0,1-1/n}$ , and  $\tilde{f}(z) = z^a f(z^b)$  belongs to  $\mathcal{Y}_{a, b/n-1}$  ( $a \in \mathbb{Z}, b \in \mathbb{N}$ ).
- $f' \in \mathcal{Y}_{\alpha,\beta}$  implies  $f \in \mathcal{Y}_{-\beta,\beta}$  for at least one primitive.
- “Most” of the first, second and fourth Painlevé transcendents belong to  $\mathcal{Y}_{\frac{1}{2}, \frac{1}{4}}$ ,  $\mathcal{Y}_{\frac{1}{2}, \frac{1}{2}}$  and  $\mathcal{Y}_{1,1}$ , respectively (for “some” solutions the second condition is violated, namely those having large zero- and pole-free regions).
  - Any first Painlevé transcendent has a primitive  $W$  which also is a first integral:  $w'^2 = 2zw + 4w^3 - 2W$ ; in “most” cases  $W \in \mathcal{Y}_{\frac{1}{4}, \frac{1}{4}}$ , although  $w'^2, zw, w^3 \in \mathcal{Y}_{\frac{3}{2}, \frac{1}{4}}$ .
  - The second Painlevé equation  $w'' = a + zw + 2w^3$  has a first integral  $W$ :  $w'^2 = 2aw + zw^2 + w^4 - W$  with  $W' = w^2$ ; since  $w^2 \in \mathcal{Y}_{1, \frac{1}{2}}$  (in “most” cases),  $W \in \mathcal{Y}_{\frac{1}{2}, \frac{1}{2}}$  follows, although  $w'^2, zw, w^4 \in \mathcal{Y}_{2, \frac{1}{2}}$ .

- Painlevé's fourth equation  $2ww'' = w'^2 + 3w^4 + 8zw^3 + 4(z^2 - a)w^2 + 2b$  also has a first integral  $W$ :  $w'^2 = w^4 + 4zw^3 + 4(z^2 - a)w^2 - 2b - 4wW$  with  $W' = w^2 + 2zw$  and  $W \in \mathcal{Y}_{1,1}$ , again only in “most” cases.

## 2. SIMPLE PROPERTIES

THEOREM 1. *Every  $f \in \mathcal{Y}_{\alpha,\beta}$  satisfies  $f^\#(z) = O(|z|^{\alpha+\beta})$ .*

PROOF. We may assume  $\alpha \geq 0$ , otherwise would replace  $f$  by  $1/f$ , noting that  $f^\# = (1/f)^\#$  and  $1/f \in \mathcal{Y}_{-\alpha,\beta}$ . For  $|h| \geq 1$  we have

$$(4) \quad f_h^\#(0) = |h|^{-\alpha-\beta} f^\#(h) \frac{1 + |f(h)|^2}{1 + |h|^{-2\alpha} |f(h)|^2} \geq |h|^{-\alpha-\beta} f^\#(h),$$

while the left hand side is bounded by Marty's Criterion. **q.e.d.**

REMARKS.

- The bound  $|z|^{\alpha+\beta}$  is sharp (not only for the Painlevé transcendents).
- It is obvious that every limit function  $\mathfrak{f} = \lim_{n \rightarrow \infty} f_{h_n}$  belongs to  $\mathcal{W}_2$ . More precisely,  $\mathfrak{f}^\#$  is bounded by  $m_f = \sup_{z \in \mathbb{C}, |h| > 1} f_h^\#(z)$ : if  $z_0$  is not a pole of  $\mathfrak{f}$ , then we have also  $f_{h_n}' \rightarrow \mathfrak{f}'$  close to  $z_0$ , hence  $\mathfrak{f}^\#(z_0) = \lim_{n \rightarrow \infty} f_{h_n}^\#(z_0) \leq m_f$ . At a pole of  $\mathfrak{f}$  we will consider  $1/\mathfrak{f}$  instead of  $\mathfrak{f}$  (more in Theorem 8).
- The limit functions of the Painlevé families  $(w_h)$  are elliptic functions.

Yosida [17] has shown that given  $f \in A_0$  and  $\epsilon > 0$  there exists some  $\delta > 0$ , such that  $\int_{|z-h| < \epsilon} f^\#(z) d(x, y) > \delta$  holds for every  $h \in \mathbb{C}$ . The analog for  $\mathcal{Y}_{\alpha,\beta}$  is Theorem 2 below. For  $\beta$  fixed,  $|h| > 1$  and  $\epsilon > 0$  we set

$$(5) \quad \Delta_\epsilon(h) = \{z : |z - h| < \epsilon |h|^{-\beta}\}.$$

THEOREM 2. *For every  $f \in \mathcal{Y}_{\alpha,\beta}$  and  $\epsilon > 0$  we have*

$$\inf_{|h| > 1} |h|^{2|\alpha|} \int_{\Delta_\epsilon(h)} f^\#(z)^2 d(x, y) > 0 \quad \text{and} \quad \inf_{|h| > 1} \sup_{z \in \Delta_\epsilon(h)} f^\#(z) |z|^{\alpha-\beta} > 0.$$

REMARK. The second inequality was proved by Gavrilo [5] for the class  $\mathcal{Y}_{0,-1}$  (which he denoted  $\mathcal{W}_1^0$ ).

PROOF. The integral in question is  $I = \int_{|w| < \epsilon} f^\#(h + h^{-\beta}w)^2 |h|^{-2\beta} d(u, v)$ . From

$$\begin{aligned} f^\#(\zeta) |h|^{-\beta} &= |h|^\alpha f_h^\#(w) \frac{1 + |h|^{-2\alpha} |f(\zeta)|^2}{1 + |f(\zeta)|^2} \\ &\geq \min\{1, |h|^{-2\alpha}\} |h|^\alpha f_h^\#(w) = |h|^{-|\alpha|} f_h^\#(w) \end{aligned}$$

( $\zeta = h + h^{-\beta}w$ ,  $|h| \geq 1$ ) follows  $|h|^{2|\alpha|} I \geq \int_{|w| < \epsilon} f_h^\#(w)^2 d(u, v)$ , and by definition of  $\mathcal{Y}_{\alpha,\beta}$  the right hand side has a positive infimum with respect to  $h$ . **q.e.d.**

THEOREM 3. *Let  $f$  be meromorphic in  $\mathbb{C}$ . Then in order that  $f \in \mathcal{Y}_{0,\beta}$  it is necessary and sufficient that*

$$f^\#(z) = O(|z|^\beta) \quad \text{and} \quad \liminf_{|h| \rightarrow \infty} \sup_{z \in \Delta_\epsilon(h)} f^\#(z)|z|^{-\beta} > 0$$

for some [all]  $\epsilon > 0$ .

PROOF. We just have to prove sufficiency. The first condition ensures that  $(f_h)$  is a normal family in  $\mathbb{C}$ , and the second guarantees that the limit functions are non-constant:  $\sup_{|z| < \epsilon} f^\#(z) > 0$ . **q.e.d.**

DEFINITION. Given  $f \in \mathcal{Y}_{\alpha,\beta}$  we denote by  $\mathcal{P}$  and  $\mathcal{Q}$  the set of non-zero poles and zeros of  $f$ , respectively (if any), and set (for the definition of  $\Delta_\epsilon$  see (5))

$$(6) \quad \mathcal{P}_\epsilon = \bigcup_{p \in \mathcal{P}} \Delta_\epsilon(p) \quad \text{and} \quad \mathcal{Q}_\epsilon = \bigcup_{q \in \mathcal{Q}} \Delta_\epsilon(q).$$

THEOREM 4. *For  $f \in \mathcal{Y}_{\alpha,\beta}$  we have*

$$\inf_{q \in \mathcal{Q}} \text{dist}(q, \mathcal{P})|q|^\beta > 0 \quad \text{and} \quad \inf_{p \in \mathcal{P}} \text{dist}(p, \mathcal{Q})|p|^\beta > 0.$$

PROOF. Take any sequence  $(q_n)$  of zeros such that  $\text{dist}(q_n, \mathcal{P})|q_n|^\beta \rightarrow \inf_{q \in \mathcal{Q}} \text{dist}(q, \mathcal{P})|q|^\beta$  and  $f_{q_n} \rightarrow f \neq \text{const}$ , locally uniformly in  $\mathbb{C}$ . Then  $f(0) = 0$  implies  $|f(z)| < 1$  on some disc  $|z| < \delta$ , hence  $\liminf_{n \rightarrow \infty} \text{dist}(q_n, \mathcal{P})|q_n|^\beta \geq \delta$  by Hurwitz' theorem, this showing that  $\inf_{q \in \mathcal{Q}} \text{dist}(q, \mathcal{P})|q|^\beta \geq \delta > 0$ . Concerning the second assertion we just note that  $1/f \in \mathcal{Y}_{-\alpha,\beta}$ , so that the notions ‘‘pole’’ and ‘‘zero’’ may be interchanged. **q.e.d.**

REMARK. We will say that the zeros and poles of  $f$  are  $\beta$ -separated. From now on it will be tacitly assumed that  $\mathcal{Q}_\epsilon \cap \mathcal{P}_\epsilon = \emptyset$ .

THEOREM 5. *Every  $f \in \mathcal{Y}_{\alpha,\beta}$  satisfies*

$$\begin{aligned} \text{(i)} \quad & |f(z)| = O(|z|^\alpha) & (z \notin \mathcal{P}_\epsilon); \\ \text{(ii)} \quad & |1/f(z)| = O(|z|^{-\alpha}) & (z \notin \mathcal{Q}_\epsilon); \\ \text{(iii)} \quad & |f(z)| \asymp |z|^\alpha & (z \notin \mathcal{P}_\epsilon \cup \mathcal{Q}_\epsilon); \\ \text{(iv)} \quad & |f'(z)/f(z)| = O(|z|^\beta) & (z \notin \mathcal{P}_\epsilon \cup \mathcal{Q}_\epsilon); \\ \text{(v)} \quad & f^\#(z) = O(|z|^{\beta-|\alpha|}) & (z \notin \mathcal{P}_\epsilon \cup \mathcal{Q}_\epsilon). \end{aligned}$$

PROOF. Let  $(h_n)$  be any sequence outside  $\mathcal{P}_\epsilon$ , such that  $f_{h_n}$  tends to  $f \neq \text{const}$ , locally uniformly in  $\mathbb{C}$ , and  $|f(h_n)||h_n|^{-\alpha}$  tends to  $M_\epsilon = \sup_{z \notin \mathcal{P}_\epsilon} |f(z)||z|^{-\alpha}$ . Then  $M_\epsilon = |f(0)|$  is finite. The second assertion follows from  $1/f \in \mathcal{Y}_{-\alpha,\beta}$ , and together we obtain  $|f(z)| \asymp |z|^\alpha$  ( $z \notin \mathcal{P}_\epsilon \cup \mathcal{Q}_\epsilon$ ). (iv) follows from  $\frac{h_n^{-\beta} f'(h_n)}{f(h_n)} \rightarrow \frac{f'(0)}{f(0)} \neq \infty$ , and from (iii) and (iv) follows  $f^\#(z) = \frac{|f'(z)|}{|f(z)|} \frac{1}{|f(z)| + \frac{1}{|f(z)|}} = O(|z|^\beta |z|^{-|\alpha|})$ , hence (v). **q.e.d.**

REMARK. The symbol  $\asymp$  has proved very useful:  $\phi(z) \asymp \psi(z)$  in some real or complex region means  $|\phi(z)| = O(|\psi(z)|)$  and  $|\psi(z)| = O(|\phi(z)|)$ .

COROLLARY 1. *Every function  $f \in \mathcal{Y}_{\alpha,\beta}$  has infinitely many zeros and poles.*

PROOF. If  $f$  had only finitely many poles, then  $f$  were rational as follows from  $f(z) = O(|z|^\alpha)$  outside  $\mathcal{P}_\epsilon$ , hence in  $|z| > R$ . **q.e.d.**

THEOREM 6. *For  $f \in \mathcal{Y}_{\alpha,\beta}$  and  $\tilde{f} \in \mathcal{Y}_{\tilde{\alpha},\beta}$  with sets of poles and zeros  $\mathcal{Q}$  and  $\tilde{\mathcal{Q}}$ , and  $\mathcal{P}$  and  $\tilde{\mathcal{P}}$ , respectively, the product  $f\tilde{f}$  belongs to  $\mathcal{Y}_{\alpha+\tilde{\alpha},\beta}$  if  $\mathcal{Q} \cup \tilde{\mathcal{Q}}$  and  $\mathcal{P} \cup \tilde{\mathcal{P}}$  are  $\beta$ -separated. In particular,  $f^m$  belongs to  $\mathcal{Y}_{m\alpha,\beta}$ .*

PROOF. The hypotheses ensure that zeros [poles] of  $f$  cannot collide with poles [zeros] of  $\tilde{f}$ , hence  $f_{h_n}\tilde{f}_{h_n} \rightarrow f\tilde{f}$ . **q.e.d.**

By Theorem 1 the zeros and poles of  $f \in \mathcal{Y}_{\alpha,\beta}$  are  $\beta$ -separated. On the other hand, zeros and poles are *equally  $\beta$ -distributed* in the following sense:

THEOREM 7. *Given  $f \in \mathcal{Y}_{\alpha,\beta}$  there exist positive numbers  $\epsilon_0$ ,  $\eta_0$ , and  $M$ , such that*

- (i) *every disc  $\Delta_{\eta_0}(z_0)$  contains at least one zero and one pole;*
- (ii) *every disc  $\Delta_{\epsilon_0}(z_0)$  contains at most  $M$  zeros (counted by multiplicities) and no pole, or at most  $M$  poles of  $f$  and no zeros.*

*In particular, the zeros and poles of  $f$  have bounded multiplicities.*

PROOF. Suppose there exist sequences  $h_n \rightarrow \infty$  and  $\eta_n \rightarrow \infty$ , such that  $\Delta_{\eta_n}(h_n)$  contains no poles (the same for zeros), while  $f_{h_n} \rightarrow \mathfrak{f} \neq \text{const}$ , locally uniformly in  $\mathbb{C}$ . Then by Hurwitz' Theorem,  $\mathfrak{f}$  is finite in every euclidian disc  $|z| < \eta_n$ , hence is an entire function, this contradicting Corollary 1. Similarly, if we assume that the pair  $(\epsilon_0, M)$  does not exist, then there exist sequences  $\epsilon_n \rightarrow 0$  and  $h_n \rightarrow \infty$ , such that  $f$  has at least  $n$  zeros (say) in  $\Delta_{\epsilon_n}(h_n)$ , while  $f_{h_n}$  tends to some non-constant function  $\mathfrak{f}$ . By Hurwitz' theorem,  $\mathfrak{f}$  has a zero at the origin of order  $\geq n$  for every  $n$ , which is absurd. Thus there exists  $\epsilon > 0$ , such that the number of zeros in  $\Delta_{\epsilon_0}(z_0)$  is bounded, uniformly with respect to  $z_0$ . Diminishing  $\epsilon_0$ , if necessary, it we may achieve by Theorem 1 that none of the discs  $\Delta_{\epsilon_0}(z_0)$  contains a pole. **q.e.d.**

REMARK. It is not hard to prove that there also exists some  $\theta_0 > 0$ , such that  $f$  assumes *every* value in *every* disc  $\Delta_{\theta_0}(z_0)$ .

THEOREM 8. [THE YOSIDA CLASS IS UNIVERSAL] *For  $f \in \mathcal{Y}_{\alpha,\beta}$  ( $\beta > -1$ ) the limit functions  $\mathfrak{f} = \lim_{h_n \rightarrow \infty} f_{h_n}$  belong to the Yosida class  $A_0 = \mathcal{Y}_{0,0}$ .*

PROOF. First of all  $\mathfrak{f}$  has bounded spherical derivative, hence  $\mathfrak{f} \in \mathcal{W}_2$  and the family  $(\mathfrak{f}_h)_{h \in \mathbb{C}}$  of translations  $\mathfrak{f}_h(z) = \mathfrak{f}(z + h)$  is normal in  $\mathbb{C}$ . Also the corresponding sets  $\mathcal{P}$  and  $\mathcal{Q}$  are 0-separated (euclidian distance between  $\mathcal{P}$  and  $\mathcal{Q}$  is positive), and equally 0-distributed: there exist positive numbers  $\epsilon_0$ ,  $\eta_0$  and  $M$ , such that every disc  $|z - h| < \eta_0$  contains at least one zero and one pole, while every disc  $|z - h| < \epsilon_0$  contains at most  $M$  poles [zeros], and no zeros [poles]. If  $(h_n)$  is any sequence tending to  $\infty$ , then the disc  $|z - h_n| < \eta_0$  contains at least one zero  $q_n$  and one pole  $p_n$ . Since  $|p_n - q_n| \geq 2\epsilon_0$ , all limit functions of  $(\mathfrak{f}_{h_n})$  also have at least one zero and one pole in  $|z| \leq 2\eta_0$ , and therefore are non-constant. **q.e.d.**

### 3. VALUE DISTRIBUTION

In this section we are concerned with the value distribution of functions  $f \in \mathcal{Y}_{\alpha,\beta}$ . For the definition of the Nevanlinna functions  $T(r, f)$ ,  $m(r, f)$  and  $N(r, f)$ , and for

basic results in Nevanlinna Theory the reader is referred to Hayman [7] and Nevanlinna [10]. From the Ahlfors-Shimizu formula

$$T(r, f) = \frac{1}{\pi} \int_0^r \int_{|z|<t} f^\#(z)^2 d(x, y) dt + O(1)$$

and Theorem 1 follows  $T(r, f) = O(r^{2(|\alpha|+\beta+1)})$ , hence  $f \in \mathcal{Y}_{\alpha, \beta}$  has order of growth

$$\varrho(f) = \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r}$$

at most  $2(|\alpha| + \beta + 1)$ . Replacing  $f$  by  $\tilde{f}(z) = z^a f(z^b)$  with  $\tilde{f} \in \mathcal{Y}_{a+b\alpha, b+\beta-1}$  yields  $\varrho(f) = \varrho(\tilde{f})/b \leq 2(|a+b\alpha| + b\beta + b)/b = 2(|a/b + \alpha| + \beta + 1)$ , and since  $\inf_{a \in \mathbb{Z}, b \in \mathbb{N}} |a/b + \alpha| = 0$ , we obtain in any case:

**THEOREM 9.** *Every  $f \in \mathcal{Y}_{\alpha, \beta}$  has order of growth  $\varrho(f) \leq 2\beta + 2$ .*

**REMARK.** For the first Painlevé transcendents the first estimate yields  $\varrho(w) \leq \frac{7}{2}$ , while the order is  $\varrho(w) = \frac{5}{2} = 2(\frac{1}{4} + 1)^{(1)}$ . Similarly we have the (sharp) estimates  $\varrho(w) \leq 3 = 2(\frac{1}{2} + 1)$  and  $\varrho(w) \leq 4 = 2(1 + 1)$  for the second and fourth Painlevé transcendents, respectively (see [15]).

**THEOREM 10.** *Every  $f \in \mathcal{Y}_{\alpha, \beta}$  ( $\beta > -1$ ) has  $\asymp r^{2\beta+2}$  zeros and poles in  $|z| < r$ :*

$$(7) \quad n(r, 0) \asymp r^{2\beta+2} \quad \text{and} \quad n(r, \infty) \asymp r^{2\beta+2}.$$

*In particular,  $f$  has order of growth  $\varrho(f) = 2\beta + 2$ .*

**REMARK.** We remind the reader that  $\phi(r) \asymp \psi(r)$  means  $\phi(r) = O(\psi(r))$  and  $\psi(r) = O(\phi(r))$  as  $r \rightarrow \infty$ .

**PROOF.** With every pole  $p$  in  $|p| < r$  we associate the disc  $\Delta_{\epsilon_0}(p)$ ; by Theorem 7 it contains at most  $M$  poles. Starting with  $p_1$  ( $|p_1| < r$ ), let  $p_2$  ( $|p_2| < r$ ) be any of the poles not contained in  $\Delta_{\epsilon_0}(p_1)$ ,  $p_3$  ( $|p_3| < r$ ) not contained in  $\Delta_{\epsilon_0}(p_1) \cup \Delta_{\epsilon_0}(p_2)$ , and so forth; we may arrange that  $|p_\nu|^{-\beta} \geq |p_{\nu+1}|^{-\beta}$  holds. Then obviously  $n(r, \infty) = O(\phi(r))$ , where  $\phi(r)$  counts how many *mutually disjoint* discs  $\Delta_{\epsilon_0/2}(p)$  may be placed in a large euclidian disc  $|z| < r + \epsilon_0 r^{-\beta}$ . The geometric answer is  $\phi(r) = O(r^{2\beta+2})$ , if  $\beta \geq 0$ , and  $\phi(r) \leq \phi(r/2) + O(r^{2\beta+2})$  if  $-1 < \beta < 0$ , which also implies  $\phi(r) = O(r^{2\beta+2})$  (consider the radii  $r = 2^k$ ). Thus

$$n(r, \infty) = O(r^{2\beta+2})$$

holds in any case. To prove the converse, we note that for  $r$  sufficiently large the annulus  $||z| - r| < \eta_0 r^{-\beta}$  contains at least  $c' r^{\beta+1}$  mutually disjoint discs of radius  $\eta_0 r^{-\beta}$ , hence also at least  $c' r^{\beta+1}$  poles. Again we have to distinguish the cases (i)  $\beta \geq 0$  and (ii)  $-1 < \beta < 0$ . Starting with  $r_1$  sufficiently large we define in case (i)  $r_k = r_{k-1} + 2\eta_0 r_{k-1}^{-\beta}$ , while in case (ii)  $r_k$  denotes the unique solution to the equation  $r_k = r_{k-1} + 2\eta_0 r_k^{-\beta}$  ( $k = 2, 3, \dots$ ); note that  $r \mapsto r - 2\eta_0 r^{-\beta}$  is increasing on  $r^{-\beta-1} < 1/(2|\beta|\eta_0)$  if  $-1 < \beta < 0$ . Then the annuli  $||z| - r_k| < \eta_0 r_k^{-\beta}$  are mutually disjoint, and each contains at least  $c' r_k^{\beta+1}$  poles of  $f$ . We claim

$$\nu_k = n(r_k, \infty) \geq 2c r_k^{2\beta+2},$$

<sup>1</sup>There is a misprint in [15]: “ $T(r, f) = 2T(r, w) + O(\log r)$ ” for  $f(z) = z^{-1}w(z^2)$ , of course, has to be replaced by “ $T(r, f) = T(r^2, w) + O(\log r)$ ”.

provided  $c$  is sufficiently small, this implying  $n(r, \infty) \geq cr^{2\beta+2}$  for  $r$  sufficiently large (note that  $r_k \rightarrow \infty$ ). Assuming  $\nu_{k-1} \geq 2cr_{k-1}^{2\beta+2}$  to be true, we obtain

$$\begin{aligned} \nu_k &\geq \nu_{k-1} + c'r_k^{\beta+1} \geq 2cr_{k-1}^{2\beta+2} + c'r_k^{\beta+1} \\ &\geq 2cr_k^{2\beta+2} - 2c(r_k - r_{k-1})(2\beta+2)r_k^{2\beta+1} + c'r_k^{\beta+1} \end{aligned}$$

by the Mean Value Theorem. In case (ii) we have  $r_k - r_{k-1} = 2\eta_0 r_k^{-\beta}$ , while in case (i)  $r_k - r_{k-1} = 2\eta_0 r_{k-1}^{-\beta} \leq 3\eta_0 r_k^{-\beta}$  holds (assuming  $r_1$  sufficiently large). We thus obtain

$$\nu_k \geq 2cr_k^{2\beta+2} + r_k^{\beta+1}[c' - 2c3\eta_0(2\beta+2)] = 2cr_k^{2\beta+2}$$

if  $c$  is chosen to satisfy  $c' - 2c3\eta_0(2\beta+2) = 0$ . Finally from  $r_k = O(r_{k-1})$  follows

$$r^{2\beta+2} = O(n(r, \infty))$$

in all cases  $\beta > -1$ . The assertion about the order of growth now follows from  $\varrho(f) \leq 2\beta+2$  on one hand, and  $T(r, f) \geq N(r, f) \asymp r^{2\beta+2}$  on the other. **q.e.d.**

From the proof we obtain:

**COROLLARY 2.** *For  $\beta > -1$  and  $c > \eta_0$ , every annulus  $||z| - r| < cr^{-\beta}$  contains  $\asymp r^{\beta+1}$  zeros [poles] of  $f \in \mathcal{Y}_{\alpha, \beta}$ .*

**THEOREM 11.** *For every  $f \in \mathcal{Y}_{\alpha, \beta}$  holds*

$$(8) \quad m(r, f) + m(r, 1/f) = \frac{1}{2\pi} \int_0^{2\pi} |\log |f(re^{i\theta})|| d\theta = O(\log r),$$

and, in particular,

$$(9) \quad T(r, f) \sim N(r, f) \sim N(r, 1/f) \asymp r^{2\beta+2}.$$

**REMARK.** For the class  $\mathcal{Y}_{0,0}$  the first assumption was proved by Favorov [3], even with  $O(1)$  instead of  $O(\log r)$ .

**PROOF.** Let  $C_r$  denote the circle  $|z| = r$ . For  $0 < \epsilon < \epsilon_0$  fixed, the contribution of  $C_r \setminus (\mathcal{Q}_\epsilon \cup \mathcal{P}_\epsilon)$  to the integral is  $O(\log r)$  by Theorem 4 (it says, among others, that  $|f(z)| \asymp |z|^\alpha$ ). Let  $K$  be a component of  $\mathcal{Q}_\epsilon$  [or  $\mathcal{P}_\epsilon$ ] that intersects the circle  $C_r$ . If  $K$  contains the zeros  $q_\mu$  ( $1 \leq \mu \leq m \leq M$ ) [or the poles  $p_\nu$  ( $1 \leq \nu \leq n \leq M$ ), but not zeros and poles simultaneously], then  $\Phi(z) = f(z) \prod_{\mu=1}^m (z - q_\mu)^{-1}$  is zero- and pole-free in  $K$  and satisfies  $|\Phi(z)| \asymp r^\alpha r^{-m\beta}$  on  $\partial K$  by Theorem 4, and also in  $K$  by the Maximum-Minimum Principle. Thus the contribution of  $C_r \cap K$  to the integral is

$$\sum_{\mu=1}^m \int_{I_r} |\log |re^{i\theta} - q_\mu|| d\theta + O(|I_r| \log r),$$

where  $I_r = \{\theta \in [0, 2\pi) : re^{i\theta} \in K\}$  and  $|I_r|$  is its linear measure. From Lemma 2 at the end of section 5 follows  $|I_r| = O(r^{-\beta-1})$ , hence

$$\int_{I_r} |\log |re^{i\theta} - q_\mu|| d\theta = O\left(\int_0^{r^{-\beta-1}} |\log(r\theta)| d\theta\right) = O(r^{-\beta-1} \log r).$$

The assertion follows from the fact, that by virtue of Corollary 2 there are at most  $O(r^{\beta+1})$  components  $K$  intersecting  $C_r$ . **q.e.d.**

THEOREM 12. For every  $f \in \mathcal{Y}_{\alpha,\beta}$  and  $c \in \mathbb{C}$  we have  $m\left(r, \frac{1}{f-c}\right) = O(\log r)$ .

REMARK. Yosida [17] proved  $m(r, 1/(f-c)) = O(r)$  for  $f \in A_0$ .

PROOF. We just note that Theorem 11 also holds for  $f-c$  instead of  $f$ . For  $\alpha \geq 0$  we have  $f-c \in \mathcal{Y}_{\alpha,\beta}$ , while  $1/f \in \mathcal{Y}_{-\alpha,\beta}$  if  $\alpha < 0$  and

$$|\log |f-c|| \leq |\log |c|| + |\log |f|| + |\log |1/f - 1/c||. \quad \text{q.e.d.}$$

THEOREM 13. The  $c$ -points ( $c \neq 0$ ) of  $f \in \mathcal{Y}_{\alpha,\beta}$  are  $\beta$ -close to the zeros, and  $\beta$ -separated from the poles if  $\alpha > 0$ , and vice versa if  $\alpha < 0$ :

$$\lim_{\zeta \rightarrow \infty, f(\zeta)=c} |\zeta|^\beta \text{dist}(\zeta, \mathcal{Q}) = 0 \quad \text{and} \quad \inf_{f(\zeta)=c} |\zeta|^\beta \text{dist}(\zeta, \mathcal{P}) > 0.$$

For  $\alpha = 0$  and any pair  $(a, b)$  the sets of  $a$ - and  $b$ -points are  $\beta$ -separated.

PROOF. The first assertion ( $\alpha > 0$ ) follows from  $f-c \in \mathcal{Y}_{\alpha,\beta}$  and Theorem 4. If  $(\zeta_n)$  denotes any sequence of  $c$ -points such that  $f_{\zeta_n} \rightarrow f \not\equiv \text{const}$ , then we have also  $\zeta_n^{-\alpha}(f(\zeta_n + \zeta_n^{-\beta}z) - c) \rightarrow f(z)$  and  $f(0) = 0$ . From Hurwitz' Theorem then follows  $|\zeta_n|^\beta \text{dist}(\zeta_n, \mathcal{Q}) \rightarrow 0$  ( $n \rightarrow \infty$ ). Finally, since  $\mathcal{Y}_{0,\beta}$  is Möbius invariant, every pair  $(a, b)$  can play the role of  $(0, \infty)$ . q.e.d.

#### 4. DERIVATIVES

The derivative of  $f_h$  is  $f'_h(z) = h^{-\alpha-\beta} f'(h + h^{-\beta}z)$ , and since the limit functions of the family  $(f_h)$  are non-rational, one might expect that  $f' \in \mathcal{Y}_{\alpha+\beta,\beta}$ . Now a trivial necessary condition for  $\phi_n \rightarrow \phi \not\equiv \text{const}$ , locally uniformly in some domain  $D$ , is that the  $a$ -points and  $b$ -points of  $\phi_n$  are locally uniformly 0-separated (separated with respect to euclidian metric in any compact subset of  $D$ ). In general,  $\phi_n \rightarrow \phi$  does not imply  $\phi'_n \rightarrow \phi'$  if  $\phi_n$  has poles, in other word, there is no Weierstrass Convergence Theorem for meromorphic functions (while the converse is true:  $\phi'_n \rightarrow \psi$  implies that  $\psi$  has a primitive  $\phi$ , and  $\phi_n \rightarrow \phi + \text{const}$ ). The obstacle that prevents  $\phi'_n$  from converging to  $\phi'$  is the existence of colliding poles of  $\phi_n$  and/or of zeros of  $\phi'_n$  colliding with poles.

LEMMA 1. Suppose that  $\phi_n$  converges to  $\phi$ , locally uniformly in  $|z| < r$ , and  $\phi$  has a pole of order  $m$  at  $z = 0$ . Then for  $\phi'_n \rightarrow \phi'$ , locally uniformly in some neighbourhood of  $z = 0$ , each of the following conditions is necessary and sufficient: there exist  $\rho > 0$  and  $n_0$ , such that for  $n \geq n_0$

- (i)  $\phi_n$  has only one pole (of order  $m$ ) in  $|z| < \rho$ ;
- (ii)  $\phi'_n$  has no zeros in  $|z| < \rho$ .

PROOF. Since  $\phi'$  has a pole of order  $m+1$  at  $z = 0$ , and no other pole and also no zero in  $|z| < 2\rho$ , it is necessary for  $\phi'_n \rightarrow \phi'$ , uniformly in some neighbourhood of  $z = 0$ , that  $\phi'_n$  ( $n \geq n_0$ ) has  $m+1$  poles (counted with multiplicities) and no zero in  $|z| < \rho$ , say. Since every pole of  $\phi_n$  of order  $\ell$  is a pole of order  $\ell+1$  of  $\phi'_n$ , this means that  $\phi_n$  has only one pole in  $|z| < \rho$ . Conversely, if  $\phi_n$  has only one pole  $b_n$  (of order  $m$ ) with  $b_n \rightarrow 0$ , then we have  $\phi_n(z) = \frac{\psi_n(z)}{(z-b_n)^m}$ ,  $\psi_n \rightarrow \psi$ ,  $\psi(0) \neq 0$ ,  $\psi'_n \rightarrow \psi'$ ,  $z\psi'(z) - m\psi(z)|_{z=0} \neq 0$ , and

$$\phi'_n(z) = \frac{(z-b_n)\psi'_n(z) - m\psi_n(z)}{(z-b_n)^{m+1}} \rightarrow \frac{z\psi'(z) - m\psi(z)}{z^{m+1}} = \phi'(z),$$



uniformly in some neighbourhood of  $z = 0$ . It remains to show that (ii) implies (i). If  $\phi_n$  has  $p > 1$  different poles in  $|z| < \rho$  of total multiplicity  $m$ , then by the Riemann-Hurwitz formula  $\phi$  has  $m - 1$  critical points close to  $z = 0$ , only  $m - p$  of them arising from multiple poles. Thus  $\phi'_n$  has  $p - 1$  zeros close to  $z = 0$ . **q.e.d.**

REMARK. In any case the sequence  $\phi'_n$  tends to  $\phi'$ , locally uniformly in  $0 < |z| < \rho$ . If (i) or (ii) is violated, then some of the poles of  $\phi'_n$  collide with zeros of  $\phi'_n$ , and in the limit multiplicities disappear as do the zeros of  $\phi'_n$ . If  $\phi_n = 1/\mathcal{P}_n$ ,  $\mathcal{P}_n$  a polynomial of degree  $m$ , the equivalence of (i) and (ii) follows from the Gauß-Lucas Theorem.

THEOREM 14. *In order that for  $f \in \mathcal{Y}_{\alpha,\beta}$  the derivative  $f'$  belongs to  $\mathcal{Y}_{\alpha+\beta,\beta}$ , each of the following conditions is necessary and sufficient:*

- (i)  $\inf_{p \in \mathcal{P}} |p|^{-\beta} \text{dist}(p, \mathcal{P} \setminus \{p\}) > 0$ ;
- (ii)  $\inf_{f'(c)=0} |c|^{-\beta} \text{dist}(c, \mathcal{P}) > 0$ .

COROLLARY 3. *If the poles of  $f \in A_0$  are 0-separated from each other, then every derivative of  $f$  also belongs to  $A_0$ .*

EXAMPLE. We construct  $f \in \mathcal{Y}_{0,0}$  such that  $f' \notin \mathcal{Y}_{0,0}$ :  $\phi(z) = \sum_{k=1}^{\infty} \frac{1}{(z - k^2)^2 - k^{-2}}$  is meromorphic in  $\mathbb{C}$ . If  $|z - k^2| \geq k/2$  holds for every  $k$ , then  $\sum_{k=2}^{\infty} Mk^{-2}$  is a convergent majorant, hence  $f(z) = o(1)$  as  $z \rightarrow \infty$  outside  $\bigcup_{k \geq 1} \{z : |z - k^2| < k/2\}$ , while in case  $|z - \ell^2| < \ell/2$  for some  $\ell$  we have  $|z - k^2| \geq k/2$  for  $k \neq \ell$  and  $f(z) = \frac{1}{(z - \ell^2)^2 - \ell^{-2}} + o(1)$  as  $z \rightarrow \infty$  by the same reason. Thus the limit functions  $\lim_{h_n \rightarrow \infty} \phi_{h_n}$  are either constants or else have the form  $(z - z_0)^{-2}$ . Then for  $f_0 \in \mathcal{Y}_{0,0}$  we have also  $f = f_0 + \phi \in \mathcal{Y}_{0,0}$ , but  $f' \notin \mathcal{Y}_{0,0}$ .

From  $f' \in \mathcal{Y}_{\alpha+\beta,\beta}$  would follow  $m(r, 1/f') = O(\log r)$ . This, however, is true anyway and provides a new proof of Theorem 12.

THEOREM 15. *Every  $f \in \mathcal{Y}_{\alpha,\beta}$  satisfies  $m(r, 1/f') = O(\log r)$ .*

REMARK. This was proved by Yosida [17] for  $f \in A_0$  with  $O(\log r)$  replaced by  $O(r)$ .

PROOF. Taking into account that  $m(r, 1/f) = O(\log r)$  and  $m(r, f) = O(\log r)$ , hence also  $m(r, f') \leq m(r, f) + O(\log r) = O(\log r)$  holds,

$$(10) \quad m(r, 1/f') = -\frac{1}{2\pi} \int_0^{2\pi} \log f^\#(re^{i\theta}) d\theta + O(\log r)$$

follows.<sup>(2)</sup> We claim that the right hand side of (10) is  $O(\log r)$ . The lower estimate follows from Theorem 1:  $-\log f^\#(z) \geq -(|\alpha| + \beta) \log |z| + O(1)$ . It remains to prove

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<sup>2</sup>More generally, Yosida [17] proved that  $2T(r, f) - N_1(r) = -\frac{1}{2\pi} \int_0^{2\pi} \log f^\#(re^{i\theta}) d\theta + O(1)$  holds, where  $N_1(r)$  “counts” the critical points of  $f$ . The following proof is straight forward:

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \log f^\#(re^{i\theta}) d\theta &= m(r, f') - m(r, 1/f') - 2m(r, f) + O(1) \\ &= -[N(r, f) + \overline{N}(r, f)] + N(r, 1/f') - 2T(r, f) + 2N(r, f) + O(1) \\ &= [N(r, f) - \overline{N}(r, f)] + N(r, 1/f') - 2T(r, f) + O(1) \\ &= N_1(r) - 2T(r, f) + O(1). \end{aligned}$$

that

$$(11) \quad -\frac{1}{2\pi} \int_0^{2\pi} \log[r^{|\alpha|-\beta} f^\#(re^{i\theta})] d\theta \leq C$$

holds. To this end we divide  $[0, 2\pi]$  into  $\asymp r^{\beta+1}$  intervals of length  $\asymp r^{-\beta-1}$ . If (11) is not true, then there exists a sequence  $r_n \rightarrow \infty$  and intervals  $I_n$  of length  $\asymp r_n^{-\beta-1}$ , such that

$$J_n = -r_n^{\beta+1} \int_{I_n} \log[r_n^{|\alpha|-\beta} f^\#(r_n e^{i\theta})] d\theta \rightarrow \infty.$$

We may assume that  $I_n = [-r_n^{-\beta-1}, r_n^{-\beta-1}]$  and  $f_{r_n} \rightarrow \mathfrak{f} \neq \text{const.}$  From

$$f_{r_n}^\#(z) = r_n^{-\alpha-\beta} f^\#(\zeta) \frac{1 + |f(\zeta)|^2}{1 + r_n^{-2\alpha} |f(\zeta)|^2} \leq r_n^{-\alpha-\beta} f^\#(\zeta) \max\{1, r_n^{2\alpha}\} = r_n^{|\alpha|-\beta} f^\#(\zeta)$$

( $\zeta = r_n e^{i\theta} = r_n + r_n^{-\beta} z$ ,  $r_n d\theta = |d\zeta| = r_n^{-\beta} |dz|$ ) then follows

$$-\log[r_n^{|\alpha|-\beta} f^\#(\zeta)] \leq -\log f_{r_n}^\#(z) \quad \text{and} \quad \limsup_{n \rightarrow \infty} J_n \leq - \int_{[-i, i]} \log \mathfrak{f}^\#(z) |dz|. \quad \text{q.e.d.}$$

## 5. SERIES AND PRODUCT DEVELOPMENTS

Since the series  $\sum_{p \in \mathcal{P}} |p|^{-s-1}$  and  $\sum_{q \in \mathcal{Q}} |q|^{-s-1}$  diverge if  $s = 2\beta + 2 = \varrho(f)$ , and converge if  $s > 2\beta + 2$ , the canonical products  $\prod_{q \in \mathcal{Q}} E(\frac{z}{q}, [2\beta] + 2)$  and  $\prod_{p \in \mathcal{P}} E(\frac{z}{p}, [2\beta] + 2)$  converge absolutely and locally uniformly;  $E(u, g) = (1 - u)e^{u+u^2/2+\dots+u^g/g}$  denotes the Weierstrass prime factor of *genus*  $g$ . Hence any  $f \in \mathcal{Y}_{\alpha, \beta}$  has the

### HADAMARD PRODUCT REPRESENTATION

$$f(z) = z^s e^{S(z)} \frac{\prod_{q \in \mathcal{Q}} E(\frac{z}{q}, m)}{\prod_{p \in \mathcal{P}} E(\frac{z}{p}, m)}$$

( $s \in \mathbb{Z}$  and  $S$  a polynomial with  $\deg S \leq m = 2 + [2\beta]$ ), and differentiation yields the

### MITTAG-LEFFLER EXPANSION

$$\frac{f'(z)}{f(z)} = \frac{s}{z} + S'(z) + \sum_{q \in \mathcal{Q}} \frac{z^m}{(z - q)q^m} - \sum_{p \in \mathcal{P}} \frac{z^m}{(z - p)p^m}.$$

If we do not insist in *absolute* convergence, then much more can be said.

**THEOREM 16.** *Suppose  $f \in \mathcal{Y}_{\alpha, \beta}$  and  $f(0) \neq 0, \infty$ . Then*

$$(12) \quad \frac{f'(z)}{f(z)} = T_{m-1}(z) + \lim_{r \rightarrow \infty} \left[ \sum_{|q| < r} \frac{z^m}{(z - q)q^m} - \sum_{|p| < r} \frac{z^m}{(z - p)p^m} \right]$$

*holds, locally uniformly in  $\mathbb{C} \setminus (\mathcal{P} \cup \mathcal{Q})$ ;  $m$  is any integer  $> \beta$ , and  $T_{m-1}$  is the  $(m-1)$ -th Taylor polynomial for  $f'/f$  at  $z = 0$ . Each zero  $q$  and pole  $p$  in the sum occurs according to its multiplicity.*

PROOF. The following technique is well-known. Let  $\Phi$  be meromorphic in the plane having simple poles  $\xi$  with residues  $\rho(\xi)$ , and assume that  $\Phi(0) \neq 0, \infty$  and  $|\Phi(z)| = O(|z|^\beta)$  holds on the circles  $|z| = r_k \rightarrow \infty$ . Then

$$(13) \quad I_k(z) = \frac{1}{2\pi i} \int_{|\zeta|=r_k} \frac{\Phi(\zeta)z^m}{(\zeta-z)\zeta^m} d\zeta = O(r_k^{\beta-m}) \rightarrow 0 \quad (k \rightarrow \infty),$$

provided  $m > \beta$ . On the other hand, the Residue Theorem yields

$$I_k(z) = \Phi(z) + \sum_{|\xi| < r_k} \frac{\rho(\xi)z^m}{(\xi-z)\xi^m} - T_{m-1}(z),$$

with  $T_{m-1}$  the  $(m-1)$ -th Taylor polynomial of  $\Phi$  at  $z=0$ , hence

$$(14) \quad \Phi(z) = T_{m-1}(z) + \lim_{k \rightarrow \infty} \sum_{|\xi| < r_k} \frac{\rho(\xi)z^m}{(z-\xi)\xi^m}.$$

This applies to  $\Phi = f'/f$  with poles  $p$  and  $q$ , if  $r_k$  can be chosen to lie outside  $\mathcal{P}_\epsilon \cup \mathcal{Q}_\epsilon$ . If, however,  $|z| = r_k$  intersects some connected component  $C$  of  $\mathcal{P}$  and/or  $\mathcal{Q}$ , we may by virtue of Lemma 2 (see the end of this section) replace the intersection  $C \cap \{z : |z| = r_k\}$  by one or more subarcs of  $\partial C$  of total length  $O(r_k^{-\beta})$ . This way we obtain the Jordan curve  $\Gamma_k$ ; it is contained in the annulus  $A_k : ||z| - r_k| < \epsilon r_k^{-\beta}$  and since there are at most  $r_k^{1+\beta}$  such components, the length of  $\Gamma_k$  is  $O(r_k)$ . To get rid of  $\Gamma_k$  and even  $r_k$  we just remark that for  $|z| < R$  and  $r_k \rightarrow \infty$  we have

$$(15) \quad \sum_{q \in A_k \cap \mathcal{Q}} \left| \frac{z^m}{(z-q)q^m} \right| + \sum_{p \in A_k \cap \mathcal{P}} \left| \frac{z^m}{(z-p)p^m} \right| = O(r_k^{-m-1} r_k^{\beta+1}) \rightarrow 0. \quad \text{q.e.d.}$$

Noting that  $\frac{z^m}{(z-\xi)\xi^m} = \frac{1}{z-\xi} + \sum_{j=0}^{m-1} \frac{z^j}{\xi^{j+1}} = \frac{d}{dz} \log E\left(\frac{z}{\xi}, m\right)$ , we obtain:

THEOREM 17. *Every  $f \in \mathcal{Y}_{\alpha,\beta}$  may be written as*

$$(16) \quad f(z) = z^s e^{S(z)} \lim_{r \rightarrow \infty} \frac{\prod_{|q| < r} E\left(\frac{z}{q}, m\right)}{\prod_{|p| < r} E\left(\frac{z}{p}, m\right)},$$

where  $m$  is any integer  $> \beta$ ,  $s \in \mathbb{Z}$ , and  $S$  is a polynomial with  $\deg S \leq m$ . Each zero  $q$  and pole  $p$  in the products occurs according to its multiplicity.

REMARK. There are, of course, also more or less complicated modifications of Theorem 16 if  $f$  has multiple poles. Since  $m > \beta$  is arbitrary, the limits

$$\lim_{r \rightarrow \infty} \left[ \sum_{|p| < r} p^{-\mu} - \sum_{|q| < r} q^{-\mu} \right]$$

exist for any integer  $\mu > m > \beta$ .

For  $\beta$  an integer, the term in brackets, the sums in (15), and also the sequence of functions  $I_k(z)$  in (13) remain uniformly bounded if we choose  $\mu = m = \beta$ , which means that in (12) and (16) we may replace  $m$  by  $\beta$  if we simultaneously replace  $r \rightarrow \infty$  by  $r_k \rightarrow \infty$  for some suitably chosen sequence  $(r_k)$ .

THEOREM 17A. *If  $\beta > -1$  is an integer, then every  $f \in \mathcal{Y}_{\alpha, \beta}$  may be written as*

$$f(z) = z^s e^{S(z)} \lim_{k \rightarrow \infty} \frac{\prod_{|q| < r_k} E\left(\frac{z}{q}, \beta\right)}{\prod_{|p| < r_k} E\left(\frac{z}{p}, \beta\right)},$$

for some suitably chosen sequence  $r_k \rightarrow \infty$ ;  $s$  is an integer and  $S$  is a polynomial with  $\deg S \leq \beta + 1$ . In particular, for  $f$  in the Yosida class  $A_0$  and also in  $\mathcal{Y}_{\alpha, 0}$  this means

$$f(z) = z^s e^{az+b} \lim_{k \rightarrow \infty} \frac{\prod_{|q| < r_k} \left(1 - \frac{z}{q}\right)}{\prod_{|p| < r_k} \left(1 - \frac{z}{p}\right)}$$

and

$$\frac{f'(z)}{f(z)} = a + \frac{s}{z} + \lim_{k \rightarrow \infty} \left[ \sum_{|q| < r_k} \frac{1}{z - q} - \sum_{|p| < r_k} \frac{1}{z - p} \right].$$

REMARK. The method of proof of Theorem 16 applies also to  $f$  itself. If  $f$  has only simple poles  $p$  with residues  $\rho(p)$  and if  $f(0) \neq \infty$ , then

$$(17) \quad f(z) = T(z) + \lim_{k \rightarrow \infty} \sum_{p \in D_k} \frac{\rho(p) z^m}{(z - p) p^m}$$

holds, locally uniformly in  $\mathbb{C} \setminus \mathcal{P}$ ;  $m$  is any integer  $> \alpha$ , and  $T$  is the  $(m-1)$ -th Taylor polynomial of  $f$  at  $z = 0$ . To get rid of  $D_k$  we need information about the residues. Let  $C_k$  be any component of  $\mathcal{P}_\epsilon$  that intersects the circle  $|z| = r_k$ . We may assume that  $C_k$  contains the poles  $p_k^{(\nu)}$  ( $1 \leq \nu \leq n \leq M_f$ ,  $p_k = p_k^{(1)}$ ), and also that  $f_{p_k} \rightarrow \mathfrak{f} \neq \text{const.}$  Then the contribution of the poles in the sequence  $(C_k)$  to  $\mathfrak{f}$  is

$$\lim_{k \rightarrow \infty} \sum_{\nu=1}^n \frac{\rho(p_k^{(\nu)}) p_k^{\beta-\alpha}}{(p_k^{(\nu)} - p_k) p_k^\beta} = P(z) \prod_{\nu=1}^n (z - a_\nu)^{-1};$$

$P$  is a polynomial of degree  $< n$ , and the numbers  $a_\nu = \lim_{k \rightarrow \infty} (p_k^{(\nu)} - p_k) p_k^\beta$  are not necessarily distinct, since  $\mathfrak{f}$  may have multiple poles. If  $a_\kappa = \dots = a_\lambda$  and  $\neq a_\mu$  else, then  $\left| \sum_{\nu=\kappa}^\lambda \rho(p_k^{(\nu)}) \right| = O(|p_k|^{\alpha-\beta})$  holds. Since there are at most  $O(r_k^{\beta+1})$  components  $C_k$ , the contribution of the annulus  $A_k$  to the sum in (17) is  $O(r_k^{\alpha-m}) \rightarrow 0$  as  $k \rightarrow \infty$ , and again we obtain

$$(18) \quad f(z) = T(z) + \lim_{r \rightarrow \infty} \sum_{|p| < r} \frac{\rho(p) z^m}{(z - p) p^m}.$$

In the particular case  $f \in A_0 = \mathcal{Y}_{0,0}$  with simple poles only, (18) holds with  $m = \alpha = 0$  and  $r = r_k$ , hence

$$f(z) = a + \lim_{k \rightarrow \infty} \sum_{|p| < r_k} \frac{\rho(p)}{z - p}.$$

We finish this section by proving a technical lemma as follows:

LEMMA 2. *Let  $C$  be any domain that consists of  $n$  discs  $\Delta_\epsilon(h_\nu)$  and intersects  $|z| = r$ . Then for  $\epsilon$  sufficiently small and  $r$  sufficiently large,  $C$  has diameter and boundary curve length  $\leq K_n \epsilon r^{-\beta}$ ; the constant  $K_n$  only depends on  $n$ .*

PROOF. We will prove by induction that there exists some  $c > 0$ , such that for  $\epsilon$  sufficiently small any domain  $C_k = \bigcup_{\kappa=1}^k \Delta_\epsilon(h_\kappa)$  ( $1 \leq k \leq n$ ) that intersects  $|z| = r$  is contained in the annulus  $A_k : ||z| - r| < 2ck\epsilon r$  with  $\text{diam } C_k \leq 2ck\epsilon r^{-\beta}$ . This is obviously true if  $k = 1$ . Assuming the assertion to be true for  $k$  discs, we consider the domain  $C_{k+1} = C_k \cup \Delta_\epsilon(h)$  satisfying  $\text{diam } C_{k+1} \leq 2ck\epsilon r^{-\beta} + 2\epsilon|h|^{-\beta}$ . From  $\Delta_\epsilon(h) \cap A_k \neq \emptyset$  then follows  $|h|^{-\beta} < cr^{-\beta}$ ,  $C_{k+1} \subset A_{k+1}$  and  $\text{diam } C_{k+1} \leq 2c(k+1)\epsilon r^{-\beta}$ . The limitations imposed on  $\epsilon$  and  $c$  are  $(1 \pm 2cn\epsilon)^{-\beta} \leq c$  and  $2cn\epsilon < 1$ . Thus the diameter of  $C$  and the length of the boundary curve of  $C$  is  $O(\epsilon r^{-\beta})$ . **q.e.d.**

## 6. THE CASE $\beta = -1$

The limit functions of the family of functions  $f_h(z) = h^{-\alpha}f(h + hz)$  have an essential singularity at  $z = -1$ , since zeros and poles accumulate there. Hence we postulate normality only in  $\mathbb{C} \setminus \{-1\}$  to define the class  $\mathcal{Y}_{\alpha, -1}$ . Apart from this it is not hard to verify that Theorems 1 [ $f^\#(z) = O(|z|^{\alpha+\beta})$ ], 2, 3, 4 [ $\beta$ -separation of  $\mathcal{P}$  and  $\mathcal{Q}$ ], 5, 6, 7 [ $\varrho(f) = 2\beta + 2$ ], 9, 11 [ $m(r, f) = O(\log r)$ ], 12, 13, 14, and 15, as well as Corollary 1 remain true also if  $\beta = -1$ . Beyond the fact  $\varrho(f) = 0$  we are looking for more detailed information on the growth of  $T(r, f)$  and  $n(r, c)$ . The analog to Theorem 10 in connection with Theorems 11 and 12 is as follows:

THEOREM 18. *Suppose  $f \in \mathcal{Y}_{\alpha, -1}$ . Then  $f^\#(z) = O(|z|^{\alpha-1})$ ,*

$$(19) \quad n(r, \infty) \asymp \log r \quad \text{and} \quad m(r, f) = O(\log r)$$

*hold; the same is true for every  $c \in \mathbb{C}$  instead of  $\infty$ . In particular we have*

$$(20) \quad T(r, f) = N(r, f) + O(\log r) \asymp \log^2 r.$$

PROOF. For  $\lambda > 1$  we consider the annuli  $A_n : \lambda^{n-1} \leq |z| < \lambda^n$ . By Theorem 7, each  $A_n$  contains at most  $O((\lambda - 1)^{-1})$  poles if  $\lambda$  is sufficiently close to 1 (according to  $\epsilon_0$  in Theorem 7), and at least one pole if  $\lambda$  is sufficiently large (according to  $\eta_0$  in Theorem 7). Thus  $n(r, \infty) \asymp \log r$  follows, and the same is true for any value  $c \in \mathbb{C}$  instead of  $c = \infty$ . **q.e.d.**

EXAMPLE. Transcendental meromorphic solutions to algebraic differential equations<sup>(3)</sup>  $w'^n = R(z, w)$  have order of growth  $\varrho \geq 1/3$  or else  $\varrho = 0$  (Bank and Kaufman [2], the author [13]). An example for the latter case is due to Bank

and Kaufman [1] (slightly modified):  $w'^2 = \frac{4w(w^2 - g_2/4)}{1 - z^2}$ . One of its solutions

satisfies  $f(\sin z) = \wp(z)$ , where  $\wp$  is the Weierstrass P-function to the differential equation  $\wp'^2 = 4\wp(\wp^2 - g_2/4)$ , and  $g_2 > 0$  is chosen in order that  $\wp$  has period lattice  $\pi(\mathbb{Z} + i\mathbb{Z})$ . The zeros  $\pm \cosh \pi(k + \frac{1}{2})$  and  $\sqrt{g_2}/2$ -points  $\pm \cosh \pi k$  are real, and the poles  $\pm i \sinh(\pi k)$  and  $-\sqrt{g_2}/2$ -points  $\pm i \sinh \pi(k + \frac{1}{2})$  are purely imaginary.

From  $f^\#(z)^2 = \frac{4|f(z)||f(z)^2 - g_2/4|}{|z^2 - 1|(1 + |f(z)|^2)^2}$  and the distribution of critical points follows

$f^\#(z) = O(|z|^{-1})$ , and  $f^\#(z) \asymp |z|^{-1}$  in  $|\arg z \pm \frac{\pi}{4}| < \frac{\pi}{4} - \epsilon$  and in  $|\arg z \pm \frac{3\pi}{4}| < \frac{\pi}{4} - \epsilon$ , hence  $f \in \mathcal{Y}_{0, -1}$  by Theorem 3. The  $k$ -th derivative of  $f$  belongs to  $f \in \mathcal{Y}_{-k, -1}$ .

Any limit function satisfies  $\mathfrak{f}'(z)^2 = -\frac{4\mathfrak{f}(z)(\mathfrak{f}(z)^2 - g_2/4)}{(z + 1)^2}$ , hence has the form  $\mathfrak{f}(z) =$

<sup>3</sup>Yosida's contribution is known as *Malmquist-Yosida Theorem* [16, 18].

$\wp(c + i \log(z + 1))$  – it is, of course, single-valued since  $\wp$  has period  $\pi$ , with essential singularity at  $z = -1$ .

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